Probing foamy spacetime with Variational Methods

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Abstract

A one-loop correction of the quasilocal energy in the Schwarzschild background, with flat space as a reference metric, is performed by means of a variational procedure in the Hamiltonian framework. We examine the graviton sector in momentum space, in the lowest possible state. An application to the black hole pair creation via the Casimir energy is presented. Implications on the foam-like scenario are discussed.

I. INTRODUCTION

It is generally accepted that if Einstein gravity has a ground state, this is represented by Flat space. Many attempts to discover a possible lower minimum different from the Flat one have been carried out and it seems nowadays that, at least at classical level, the lowest energy configuration state with $E = 0$ is attributed to Flat space [1]. From the quantum mechanical point of view we can obtain information on the stability of the spacetime by

means of saddle point methods: if a lower state exists, this can be reached by quantum tunneling. As a consequence a negative mode will appear in the Lichnerowicz operator. Actually a tunneling process was discovered by *Gross, Perry* and *Yaffe* [2], but it involves a flat space having a temperature $T \neq 0$, which means that we have not considered as initial state the correct "*vacuum*". Thus also this result seems to corroborate the stability of flat space under quantum fluctuation via nucleation of black holes. Recently a different mechanism involving both a topology change and a black hole pair production has been considered [3]. In this paper we are investigating the possibility of computing such effects in a Hamiltonian approach in presence of quasilocal energy. To this purpose we begin by fixing the background manifold $\mathcal{M}$ which is represented by the Schwarzschild spacetime. In particular we are interested in a constant time section of $\mathcal{M}$, best known as Einstein-Rosen bridge with wormhole topology $R^1 \times S^2$, defining a bifurcation surface which divides Σ into two parts denoted by Σ_+ and Σ_-. Our purpose is to consider perturbations at Σ, which naturally define quantum fluctuations of the Einstein-Rosen bridge and evaluate quasilocal energy corrections to one-loop with the assumption of neglecting quantum excitations on the boundaries, motivated by the fact that in asymptotic spacelike directions, quasilocal energy approaches the $\mathcal{ADM}$ term defined in the asymptotic region, whose quantum fluctuations are unphysical. Nevertheless this approximation is not valid when we consider finite distance located boundaries and multi-wormholes configuration [4,5]. In this context, we will examine the effect of perturbations on the possible "*ground state*" by means of a variational approach applied on gaussian wave functional. Indeed, if we discover the existence of negative modes in this approximation, it is quite reasonable to think of a tunneling process which moves spacetime from the false vacuum to the true one. The rest of the paper is structured as follows, in section II, we have borrowed from Refs. [5,6] the general expressions for Hamiltonian and quasilocal energy, in section III, we analyze the stationary Schrödinger equation coming from the perturbed wormhole metric and we give some of the basic rules to perform the functional integration for the Hamiltonian approximated to second order, in section IV, we analyze the spin-2 operator or the operator acting on transverse traceless tensors. We

summarize and conclude in section V.

II. QUASILOCAL ENERGY AND ENERGY DENSITY CALCULATION IN SCHRÖDINGER REPRESENTATION

Although it is not necessary for the forthcoming discussions, let us consider the maximal analytic extension of the Schwarzschild metric, i.e., the Kruskal manifold whose spatial slices Σ represent Einstein-Rosen bridges with wormhole topology $S^2 \times R^1$. Following Ref. [5], the complete manifold $\mathcal{M}$ can be taken as a model for an eternal black hole composed of two wedges $\mathcal{M}_+$ and $\mathcal{M}_-$ located in the right and left sectors of a Kruskal diagram. The hypersurface Σ is divided in two parts Σ_+ and Σ_- by a bifurcation two-surface S_0. On Σ we can write the gravitational Hamiltonian

$$H_p = H - H_0 = \int_\Sigma d^3x (N\mathcal{H} + N^i\mathcal{H}_i)$$

$$+ \frac{1}{\kappa}\int_{S_+} d^2x N\sqrt{\sigma}\left(k - k^0\right) - \frac{1}{\kappa}\int_{S_-} d^2x N\sqrt{\sigma}\left(k - k^0\right), \tag{1}$$

where $\kappa = 8\pi G$. The Hamiltonian has both volume and boundary contributions. The volume part involves the Hamiltonian and momentum constraints

$$\mathcal{H} = (2\kappa)\, G_{ijkl}\frac{\pi^{ij}\pi^{kl}}{\sqrt{^3g}} - \sqrt{^3g}R/\left(2\kappa\right) = 0,$$

$$\mathcal{H}_i = -2\pi^j_{i|j} = 0, \tag{2}$$

where $G_{ijkl} = \frac{1}{2}\left(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}\right)$ is the supermetric and R denotes the scalar curvature of the surface Σ. The volume part of the Hamiltonian (1) is zero when the Hamiltonian and momentum constraints are imposed. However, for the flat and the Schwarzschild space, constraints are immediately satisfied, then in this context the total Hamiltonian reduces to

$$H_p = \frac{1}{\kappa}\int_{S_+} d^2x N\sqrt{\sigma}\left(k - k^0\right) - \frac{1}{\kappa}\int_{S_-} d^2x N\sqrt{\sigma}\left(k - k^0\right). \tag{3}$$

Quasilocal energy is defined as the value of the Hamiltonian that generates unit time translations orthogonal to the two-dimensional boundaries, i.e.

$$E_{\rm tot} = H_{\rm quasilocal} = H_+ - H_- = E_+ - E_-,$$

$$E_+ = \frac{1}{\kappa}\int_{S_+} d^2x\sqrt{\sigma}\left(k - k^0\right)$$

$$E_- = -\frac{1}{\kappa}\int_{S_-} d^2x\sqrt{\sigma}\left(k - k^0\right). \tag{4}$$

where $|N| = 1$ at both S_+ and S_-. E_{tot} is the quasilocal energy of a spacelike hypersurface $\Sigma = \Sigma_+ \cup \Sigma_-$ bounded by two boundaries ${}^3S_+$ and ${}^3S_-$ located in the two disconnected regions M_+ and M_- respectively. We have included the subtraction terms k^0 for the energy. k^0 represents the trace of the extrinsic curvature corresponding to embedding in the two-dimensional boundaries ${}^2S_+$ and ${}^2S_-$ in three-dimensional Euclidean space. Following the same scheme of the boundary subtraction procedure, we would like to discuss the possibility of generalizing such a procedure. To this end, by looking at the Hamiltonian structure, we see that there are two classical constraints

$$\begin{cases} \mathcal{H} & = 0 \\ \mathcal{H}^i & = 0 \end{cases}, \tag{5}$$

which are satisfied both by the Schwarzschild and Flat metric and two *quantum* constraints

$$\begin{cases} \mathcal{H}\tilde{\Psi} & = 0 \\ \mathcal{H}^i\tilde{\Psi} & = 0 \end{cases}. \tag{6}$$

$\mathcal{H}\tilde{\Psi} = 0$ is known as the *Wheeler-DeWitt* equation (WDW). Nevertheless, we are interested in assigning a meaning to

$$\frac{\left\langle \Psi \left| H^{\rm Schw.} - H^{\rm Flat} \right| \Psi \right\rangle}{\langle \Psi|\Psi\rangle} + \frac{\langle \Psi \left| H_{\rm quasilocal} \right| \Psi\rangle}{\langle \Psi|\Psi\rangle}, \tag{7}$$

where Ψ is a wave functional whose structure will be determined later and $H^{\rm Schw.}\left(H^{\rm Flat}\right)$ is the total Hamiltonian referred to the different spacetimes. Note that the first term of

(1) is simply the extension of the quasilocal energy subtraction procedure generalized to the volume term. Note also that if the expectation value is calculated on the wave functional solution of the WDW equation, we obtain only the boundary contribution. Then to give meaning to (1), we adopt the semiclassical strategy of the WKB expansion. By observing that the kinetic part of the Super Hamiltonian is quadratic in the momenta, we expand the three-scalar curvature $\int d^3x\sqrt{g}R^{(3)}$ up to quadratic order and we get

$$\int d^3x \left[-\frac{1}{4}h\triangle h+\frac{1}{4}h^{li}\triangle h_{li}-\frac{1}{2}h^{ij}\nabla_l\nabla_i h^l_j+\frac{1}{2}h\nabla_l\nabla_i h^{li}-\frac{1}{2}h^{ij}R_{ia}h^a_j+\frac{1}{2}hR_{ij}h^{ij}\right], \tag{8}$$

where h is the trace of h_{ij}. On the other hand, following the usual WKB expansion, we will consider $\tilde{\Psi}\simeq C\exp(iS)$. In this context, the approximated wave functional will be substituted by a *trial wave functional* according to the variational approach we would like to implement as regards this problem.

III. GAUSSIAN WAVE FUNCTIONAL AND ENERGY DENSITY CALCULATION IN SCHRÖDINGER REPRESENTATION

To actually make such calculations, we need an orthogonal decomposition for both π_{ij} and h_{ij} to disentangle gauge modes from physical deformations. We define the inner product

$$\langle h,k\rangle := \int_\Sigma \sqrt{g}G^{ijkl}h_{ij}(x)\,k_{kl}(x)\,d^3x, \tag{9}$$

by means of the inverse WDW metric G_{ijkl}, to have a metric on the space of deformations, i.e. a quadratic form on the tangent space at h, with

$$G^{ijkl}=(g^{ik}g^{jl}+g^{il}g^{jk}-2g^{ij}g^{kl}). \tag{10}$$

The inverse metric is defined on co-tangent space and it assumes the form

$$\langle p,q\rangle := \int_\Sigma \sqrt{g}G_{ijkl}p^{ij}(x)\,q^{kl}(x)\,d^3x, \tag{11}$$

so that

$$G^{ijnm}G_{nmkl}=\frac{1}{2}\left(\delta_k^i\delta_l^j+\delta_l^i\delta_k^j\right). \tag{12}$$

Note that in this scheme the "inverse metric" is actually the WDW metric defined on phase space. Now, we have the desired decomposition on the tangent space of 3-metric deformations [7,8]:

$$h_{ij}=\frac{1}{3}hg_{ij}+(L\xi)_{ij}+h_{ij}^{\perp} \tag{13}$$

where the operator L maps ξ_i into symmetric tracefree tensors

$$(L\xi)_{ij}=\nabla_i\xi_j+\nabla_j\xi_i-\frac{2}{3}g_{ij}(\nabla\cdot\xi). \tag{14}$$

Then the inner product between three-geometries becomes

$$\langle h,h\rangle:=\int_\Sigma\sqrt{g}G^{ijkl}h_{ij}(x)\,h_{kl}(x)\,d^3x=$$

$$\int_\Sigma\sqrt{g}\left[-\frac{2}{3}h^2+(L\xi)^{ij}(L\xi)_{ij}+h^{ij\perp}h_{ij}^{\perp}\right]. \tag{15}$$

With the orthogonal decomposition in hand we can define a "*Vacuum Trial State*"

$$\Psi\left[h_{ij}(\overrightarrow{x})\right]=\mathcal{N}\exp\left\{-\frac{1}{4l_p^2}\left[\left\langle hK^{-1}h\right\rangle_{x,y}^{\perp}+\left\langle(L\xi)\,K^{-1}(L\xi)\right\rangle_{x,y}^{\|}+\left\langle hK^{-1}h\right\rangle_{x,y}^{Trace}\right]\right\}, \tag{16}$$

which will be used as a probe for the gravitational ground state. This particular expression is useful because the functional can be represented as a product of three functionals defined on the decomposed tensor field

$$\Psi\left[h_{ij}(\overrightarrow{x})\right]=\mathcal{N}\Psi\left[h_{ij}^{\perp}(\overrightarrow{x})\right]\Psi\left[(L\xi)_{ij}\right]\Psi\left[\frac{1}{3}g_{ij}h(\overrightarrow{x})\right]. \tag{17}$$

$h_{ij}^{\perp}$ is the tracefree-transverse part of the $3D$ quantum field, $(L\xi)_{ij}$ is the longitudinal part and finally h is the trace part of the same field. $\langle\cdot,\cdot\rangle_{x,y}$ denotes space integration and K^{-1} is the inverse propagator containing variational parameters. The main reason for a similar "*Ansatz*" comes from the observation that the quadratic part in the momenta of the Hamiltonian decouples in the same way of eq.(15). Note that the decomposition related to

the momenta is independent of the choice of the functional. To calculate the energy density, we need to know the action of some basic operators on $\Psi\left[h_{ij}\right]$. The action of the operator h_{ij} on $\left|\Psi\right\rangle = \Psi\left[h_{ij}\right]$ is realized by

$$h_{ij}\left(x\right)\left|\Psi\right\rangle = h_{ij}\left(\overrightarrow{x}\right)\Psi\left[h_{ij}\right]. \tag{18}$$

The action of the operator π_{ij} on $\left|\Psi\right\rangle$, in general, is

$$\pi_{ij}\left(x\right)\left|\Psi\right\rangle = -i\frac{\delta}{\delta h_{ij}\left(\overrightarrow{x}\right)}\Psi\left[h_{ij}\right]. \tag{19}$$

The inner product is defined by the functional integration:

$$\left\langle\Psi_1 \mid \Psi_2\right\rangle = \int\left[\mathcal{D}h_{ij}\right]\Psi_1^*\left\{h_{ij}\right\}\Psi_2\left\{h_{kl}\right\}, \tag{20}$$

and the energy eigenstates satisfy the stationary Schrödinger equation:

$$\int d^3x\mathcal{H}\left\{-i\frac{\delta}{\delta h_{ij}\left(\overrightarrow{x}\right)}, h_{ij}\left(\overrightarrow{x}\right)\right\}\Psi\left\{h_{ij}\right\} = E\Psi\left\{h_{ij}\right\}, \tag{21}$$

where $\mathcal{H}\left\{-i\frac{\delta}{\delta h_{ij}(x)}, h_{ij}\left(x\right)\right\}$ is the Hamiltonian density. Note that the previous equation in the general context of Einstein gravity is devoid of meaning, because of the constraints. However in the semiclassical context, we can give a meaning to eq.(21), where a *semiclassical time* is introduced in the same manner of Refs. [9,10]. There, a Schrödinger equation of the form

$$i\frac{\partial\Psi^{\perp}}{\partial t} = H_{|2}\Psi^{\perp} \tag{22}$$

is recovered by the WDW equation approximated to second order for a perturbed minisuperspace Friedmann model without boundary terms. When asymptotically flat boundary terms are present we have to take account of such contributions in the WKB expansion such as in Ref. [11]. However in this paper only gravitational transverse-traceless modes are considered on the fixed curved background and $\Psi^{\perp}$ is substituted by a trial wave functional. To further proceed, instead of solving (21), which is of course impossible, we can formulate the same problem by means of a variational principle. We demand that

$$\frac{\left\langle \Psi \mid H_{|2} \mid \Psi \right\rangle}{\langle \Psi \mid \Psi \rangle} = \frac{\int \left[\mathcal{D}g_{ij}^{\perp}\right] \int d^3x \Psi^* \left\{g_{ij}^{\perp}\right\} \mathcal{H}_{|2} \Psi \left\{g_{kl}^{\perp}\right\}}{\int \left[\mathcal{D}g_{ij}^{\perp}\right] \mid \Psi \left\{g_{ij}^{\perp}\right\} \mid^2} \tag{23}$$

be stationary against arbitrary variations of $\Psi \{h_{ij}\}$. The form of $\left\langle \Psi \mid H_{|2} \mid \Psi \right\rangle$ can be computed as follows. We define normalized mean values

$$\bar{g}_{ij}^{\perp}(\overrightarrow{x}) = \frac{\int \left[\mathcal{D}g_{ij}^{\perp}\right] \int d^3x g_{ij}^{\perp}(\overrightarrow{x}) \mid \Psi \left\{g_{ij}^{\perp}\right\} \mid^2}{\int \left[\mathcal{D}g_{ij}^{\perp}\right] \mid \Psi \left\{g_{ij}^{\perp}\right\} \mid^2}, \tag{24}$$

$$\bar{g}_{ij}^{\perp}(\overrightarrow{x})\ \ \bar{g}_{kl}^{\perp}(\overrightarrow{y}) + K_{ijkl}^{\perp}(\overrightarrow{x}, \overrightarrow{y}) \tag{25}$$

$$= \frac{\int \left[\mathcal{D}g_{ij}^{\perp}\right] \int d^3x g_{ij}^{\perp}(\overrightarrow{x})\, g_{kl}^{\perp}(\overrightarrow{y}) \mid \Psi \left\{g_{ij}^{\perp}\right\} \mid^2}{\int \left[\mathcal{D}g_{ij}^{\perp}\right] \mid \Psi \left\{g_{ij}^{\perp}\right\} \mid^2}. \tag{26}$$

It follows that, by defining $h_{ij}^{\perp} = g_{ij} - \bar{g}_{ij}$, we have

$$\int \left[\mathcal{D}h_{ij}^{\perp}\right] h_{ij}^{\perp}(\overrightarrow{x}) \mid \Psi \left\{h_{ij}^{\perp} + \bar{g}_{ij}^{\perp}\right\} \mid^2 = 0 \tag{27}$$

and

$$\int \left[\mathcal{D}h_{ij}^{\perp}\right] \int d^3x h_{ij}^{\perp}(\overrightarrow{x})\, h_{kl}^{\perp}(\overrightarrow{y}) \mid \Psi \left\{h_{ij}^{\perp} + \bar{g}_{ij}^{\perp}\right\} \mid^2 =$$

$$K_{ijkl}^{\perp}(\overrightarrow{x}, \overrightarrow{y}) \int \left[\mathcal{D}h_{ij}^{\perp}\right] \mid \Psi \left\{h_{ij}^{\perp} + \bar{g}_{ij}^{\perp}\right\} \mid^2 . \tag{28}$$

Nevertheless, the application of the variational principal on arbitrary wave functional does not improve the situation described by the eq.(21). To this purpose, we give to the trial wave functional the form

$$\Psi\left[h_{ij}^{\perp}\right] = \mathcal{N} \exp\left\{-\frac{1}{4l_p^2} \left\langle (g - \overline{g})\, K^{-1} (g - \overline{g}) \right\rangle_{x,y}^{\perp}\right\}. \tag{29}$$

We immediately conclude that

$$\left\langle \Psi | \pi_{ij}^{\perp}(\overrightarrow{x}) | \Psi \right\rangle = 0 \tag{30}$$

where $\pi_{ij}^{\perp}$ is the TT momentum. In Appendix B, we will show that

$$\left\langle \Psi | \pi_{ij}^{\perp}\left(\overrightarrow{x}\right) \pi_{kl}^{\perp}\left(\overrightarrow{y}\right) | \Psi \right\rangle = \frac{1}{4} K_{ijkl}^{-1}\left(\overrightarrow{x}, \overrightarrow{y}\right). \tag{31}$$

Choice (29) is related to the form of the Hamiltonian approximated to quadratic order in the metric deformations. Indeed, up to this order we have a harmonic oscillator whose ground state has a Gaussian form. By means of decomposition (13), we extract the TT sector contribution in the previous expression. Moreover, the functional representation (17) eliminates every interaction between gauge and the other terms. Then for the TT sector (spin-two), one gets

$$\int_{\Sigma} d^3x \sqrt{g} R^{(3)} \simeq \frac{1}{4l_p^2} \int_{\Sigma} d^3x \sqrt{g} \left[h^{\perp ij} \left(\triangle_2\right)_j^a h_{ia}^{\perp} - 2hR_{ij}h^{\perp ij} \right], \tag{32}$$

where $\left(\triangle_2\right)_j^a := -\triangle \delta_j^a + 2R_j^a$. The latter term disappears because the gaussian integration does not mix the components. Then by collecting together eq.(32) and eq.(31), one obtains the one-loop-like Hamiltonian form for TT deformations

$$H^{\perp} = \frac{1}{4l_p^2} \int_{\Sigma} d^3x \sqrt{g} G^{ijkl} \left[K^{-1\perp}\left(x, x\right)_{ijkl} + \left(\triangle_2\right)_j^a K^{\perp}\left(x, x\right)_{iakl} \right]. \tag{33}$$

The propagator $K^{\perp}\left(x, x\right)_{iakl}$ comes from a functional integration and it can be represented as

$$K^{\perp}\left(\overrightarrow{x}, \overrightarrow{y}\right)_{iakl} := \sum_{N} \frac{h_{ia}^{\perp}\left(\overrightarrow{x}\right) h_{kl}^{\perp}\left(\overrightarrow{y}\right)}{2\lambda_N\left(p\right)}, \tag{34}$$

where $h_{ia}^{\perp}\left(\overrightarrow{x}\right)$ are the eigenfunctions of $\triangle_{2j}^{a}$ and $\lambda_N\left(p\right)$ are infinite variational parameters.

IV. THE SPECTRUM OF THE SPIN-2 OPERATOR AND THE EVALUATION OF THE ENERGY DENSITY IN MOMENTUM SPACE

The Spin-two operator is defined by

$$\left(\triangle_2\right)_j^a := -\triangle \delta_j^a + 2R_j^a \tag{35}$$

where $\triangle$ is the curved Laplacian (Laplace-Beltrami operator) on a Schwarzschild background and R_j^a is the mixed Ricci tensor whose components are:

$$R_j^a = diag\left\{\frac{-2m}{r^3}, \frac{m}{r^3}, \frac{m}{r^3}\right\}, \tag{36}$$

where $2m = 2MG$. This operator is similar to the Lichnerowicz operator provided that we substitute the Riemann tensor with the Ricci tensor. This is essentially due to the fact that the Riemann tensor in three-dimensions is a linear combination of the Ricci tensor. In (37) the Ricci tensor acts as a potential on the space of TT tensors; for this reason we are led to study the following eigenvalue equation

$$\left(-\triangle\delta_j^a + 2R_j^a\right) h_a^i = E^2 h_j^i \tag{37}$$

where E^2 is the eigenvalue of the corresponding equation. In doing so, we follow Regge and Wheeler in analyzing the equation as modes of definite frequency, angular momentum and parity. We can specialize to the case with $M = 0$ without altering the contribution to the total energy because of the spherical symmetry of the problem. We recall that L is the quantum number corresponding to the square of angular momentum and M is the quantum number corresponding to the projection of the angular momentum on the z-axis. In this case, Regge-Wheeler decomposition [14] shows that the even-parity three-dimensional perturbation is

$$h_{ij}^{even}\left(r,\vartheta,\phi\right) = diag\left[H\left(r\right)\left(1-\frac{2m}{r}\right)^{-1}, r^2K\left(r\right), r^2\sin^2\vartheta K\left(r\right)\right] Y_{l0}\left(\vartheta,\phi\right). \tag{38}$$

Representation (38) shows a gravitational perturbation decoupling. For a generic value of the angular momentum L, one gets

$$\left\{\begin{array}{l} -\triangle_l H\left(r\right) - \frac{4m}{r^3}H\left(r\right) = E_l^2 H\left(r\right) \\ \\ -\triangle_l K\left(r\right) + \frac{2m}{r^3}K\left(r\right) = E_l^2 K\left(r\right) \\ \\ -\triangle_l K\left(r\right) + \frac{2m}{r^3}K\left(r\right) = E_l^2 K\left(r\right). \end{array}\right. \tag{39}$$

The Laplacian restricted to Σ can be written as

$$\triangle = \left(1-\frac{2m}{r}\right)\frac{d^2}{dr^2} + \left(\frac{2r-3m}{r^2}\right)\frac{d}{dr} - \frac{l\left(l+1\right)}{r^2}. \tag{40}$$

Defining reduced fields

$$H(r) = \frac{h(r)}{r}; \qquad K(r) = \frac{k(r)}{r}, \tag{41}$$

and passing to the proper geodesic distance from the *throat* of the bridge defined by

$$dx = \pm \frac{dr}{\sqrt{1 - \frac{2M}{r}}}, \tag{42}$$

whose integrated form is

$$x = 2m \left\{ \sqrt{\frac{r}{2m}} \sqrt{\frac{r}{2m} - 1} + \ln \left(\sqrt{\frac{r}{2m}} + \sqrt{\frac{r}{2m} - 1} \right) \right\}, \tag{43}$$

the system (39) becomes

$$\begin{cases} -\frac{d^2}{dx^2} h(x) - V^-(x) h(x) = E_l^2 h(x) \\ \\ -\frac{d^2}{dx^2} k(x) + V^+(x) k(x) = E_l^2 k(x) \\ \\ -\frac{d^2}{dx^2} k(x) + V^+(x) k(x) = E_l^2 k(x) \end{cases} \tag{44}$$

with

$$V(x) = \frac{l(l+1)}{r^2(x)} \mp \frac{3m}{r(x)^3}. \tag{45}$$

This new variable represents the proper geodesic distance from the wormhole throat such that

$$\text{when } r \longrightarrow \infty,\ x \simeq r\ ,\ V(x) \longrightarrow 0$$

$$\text{when } r \longrightarrow r_0,\ x \simeq 0,\ \ V^{\mp}(x) \longrightarrow \frac{l(l+1)}{r_0^2} \mp \frac{3m}{r_0^3} = const, \tag{46}$$

where r_0 satisfies the condition $r_0 > 2m$. The solution of (44), in both cases (flat and curved one) is the spherical Bessel function of the first kind

$$j_0(px) = \sqrt{\frac{2}{\pi}} \sin(px) \tag{47}$$

This choice is dictated by the requirement that

$$h(x), k(x) \to 0 \qquad \text{when} \qquad x \to 0 \ \left(\text{alternatively } r \to 2m\right). \tag{48}$$

Then

$$K(x,y) = \frac{j_0(px)\, j_0(py)}{2\lambda} \cdot \frac{1}{4\pi} \tag{49}$$

Substituting (49) in (33) one gets (after normalization in spin space and after a rescaling of the fields in such a way as to absorb l_p^2)

$$E(m,\lambda) = \frac{V}{2\pi^2} \sum_{l=0}^{\infty} \sum_{i=1}^{2} \int_0^{\infty} dp p^2 \left[\lambda_i(p) + \frac{E_i^2(p,m,l)}{\lambda_i(p)}\right] \tag{50}$$

where

$$E_{1,2}^2(p,m,l) = p^2 + \frac{l(l+1)}{r_0^2} \mp \frac{3m}{r_0^3}, \tag{51}$$

$\lambda_i(p)$ are variational parameters corresponding to the eigenvalues for a (graviton) spin-two particle in an external field and V is the volume of the system.

By minimizing (50) with respect to $\lambda_i(p)$ one obtains $\overline{\lambda}_i(p) = \left[E_i^2(p,m,l)\right]^{\frac{1}{2}}$ and

$$E\left(m, \overline{\lambda}\right) = \frac{V}{2\pi^2} \sum_{l=0}^{\infty} \sum_{i=1}^{2} \int_0^{\infty} dp 2 \sqrt{E_i^2(p,m)} \tag{52}$$

with

$$\text{with } p^2 + \frac{l(l+1)}{r_0^2} > \frac{3m}{r_0^3}.$$

Thus, in presence of the curved background, we get

$$E(m) = \frac{V}{2\pi^2} \frac{1}{2} \sum_{l=0}^{\infty} \int_0^{\infty} dp p^2 \left(\sqrt{p^2 + c_-^2} + \sqrt{p^2 + c_+^2}\right) \tag{53}$$

where

$$c_{\mp}^2 = \frac{l(l+1)}{r_0^2} \mp \frac{3m}{r_0^3},$$

while when we refer to the flat space, in the spirit of the subtraction procedure, we have $m = 0$ and $c^2 = \frac{l(l+1)}{r_0^2}$. Then

$$E\left(0\right)=\frac{V}{2\pi^{2}}\frac{1}{2}\sum_{l=0}^{\infty}\int_{0}^{\infty}dpp^{2}\left(2\sqrt{p^{2}+c^{2}}\right) \tag{54}$$

Now, we are in position to compute the difference between (53) and (54). Since we are interested in the UV limit, we have

$$\Delta E\left(m\right)=E\left(m\right)-E\left(0\right)$$

$$=\frac{V}{2\pi^{2}}\frac{1}{2}\sum_{l=0}^{\infty}\int_{0}^{\infty}dpp^{2}\left[\sqrt{p^{2}+c_{-}^{2}}+\sqrt{p^{2}+c_{+}^{2}}-2\sqrt{p^{2}+c^{2}}\right]$$

$$=\frac{V}{2\pi^{2}}\frac{1}{2}\sum_{l=0}^{\infty}\int_{0}^{\infty}dpp^{3}\left[\sqrt{1+\left(\frac{c_{-}}{p}\right)^{2}}+\sqrt{1+\left(\frac{c_{+}}{p}\right)^{2}}-2\sqrt{1+\left(\frac{c}{p}\right)^{2}}\right] \tag{55}$$

and for $p^{2}>>c_{\mp}^{2},c^{2}$, we obtain

$$\frac{V}{2\pi^{2}}\frac{1}{2}\sum_{l=0}^{\infty}\int_{0}^{\infty}dpp^{3}\left[1+\frac{1}{2}\left(\frac{c_{-}}{p}\right)^{2}-\frac{1}{8}\left(\frac{c_{-}}{p}\right)^{4}+1+\frac{1}{2}\left(\frac{c_{+}}{p}\right)^{2}-\frac{1}{8}\left(\frac{c_{+}}{p}\right)^{4}\right.$$

$$\left.-2-\left(\frac{c}{p}\right)^{2}-\frac{1}{4}\left(\frac{c}{p}\right)^{4}\right]=-\frac{V}{2\pi^{2}}\frac{c^{4}}{8}\int_{0}^{\infty}\frac{dp}{p}. \tag{56}$$

We will use a cut-off Λ to keep under control the UV divergence

$$\int_{0}^{\infty}\frac{dp}{p}\sim\int_{0}^{\frac{\Lambda}{c}}\frac{dx}{x}\sim\ln\left(\frac{\Lambda}{c}\right), \tag{57}$$

where $\Lambda\leq m_{p}$. Thus $\Delta E\left(m\right)$ for high momenta becomes

$$\Delta E\left(m\right)\sim-\frac{V}{2\pi^{2}}\frac{c^{4}}{16}\ln\left(\frac{\Lambda^{2}}{c^{2}}\right)=-\frac{V}{2\pi^{2}}\left(\frac{3m}{r_{0}^{3}}\right)^{2}\frac{1}{16}\ln\left(\frac{r_{0}^{3}\Lambda^{2}}{3m}\right). \tag{58}$$

Remark It is known that at one-loop level Gravity is renormalizable only in flat space. In a dimensional regularization scheme its contribution to the action is, on shell, proportional to the Euler character of the manifold that is nonzero for the Schwarzschild instanton. Although in our approach we are working with sections of the original manifold to deal with these divergences one must introduce a regulator that indeed appears in the contribution of energy density, which we presume should be taken of order the Planck mass $m_{P}\sim\sqrt{G^{-1}}$ in the

same spirit of Ref. [2]. This agrees with Wheeler's point of view concerning the gravitational vacuum fluctuations, where a natural cut-off is introduced: the Planck length $l_p \sim \sqrt{G}$ [15]. At this point we can compute the total energy, namely the classical contribution plus the quantum correction up to second order. Recalling the expression for quasilocal energy

$$E_{\text{quasilocal}} = \frac{1}{\kappa}\int_{S_+} d^2x\sqrt{\sigma}\left(k-k^0\right) - \frac{1}{\kappa}\int_{S_-} d^2x\sqrt{\sigma}\left(k-k^0\right), \tag{59}$$

and by using the expression of the trace

$$k = -\frac{1}{\sqrt{h}}\left(\sqrt{h}n^{\mu}\right)_{,\mu}, \tag{60}$$

we obtain at either boundary that

$$k = \frac{-2r_{,y}}{r}, \tag{61}$$

where we have assumed that the function $r_{,y}$ is positive for S_+ and negative for S_-. The trace associated with the subtraction term is taken to be $k^0 = -2/r$ for B_+ and $k^0 = 2/r$ for B_-. Then the quasilocal energy with subtraction terms included is

$$E_{\text{quasilocal}} = E_+ - E_- = \left(r\left[1-\left|r_{,y}\right|\right]\right)_{y=y_+} - \left(r\left[1-\left|r_{,y}\right|\right]\right)_{y=y_-}. \tag{62}$$

Note that the total quasilocal energy is zero for boundary conditions symmetric with respect to the bifurcation surface S_0. The energy E_+ (E_-) tends to the $\mathcal{ADM}$ mass M [12] whenever the boundary B_+ (B_-) tends to right (left) spatial infinity. As an illustration, consider the case when the boundary B_+ is located at right-hand infinity ($y_+ = +\infty$) and the boundary B_- is located at y_-. The total energy for the stable modes is

$$M - r\left[1-\left(1-\frac{2MG}{r}\right)^{\frac{1}{2}}\right] - \frac{V}{2\pi^2}\left(\frac{3MG}{r_0^3}\right)^2\frac{1}{16}\ln\left(\frac{r_0^3\Lambda^2}{3MG}\right), \tag{63}$$

while the total square energy for the unstable modes is

$$E^2 = -\frac{a^2}{8\left(MG\right)^2}$$

with $a^2 = 0,242$ and will be computed in the appendix C. Then the one loop total energy is

$$M-r\left[1-\left(1-\frac{2MG}{r}\right)^{\frac{1}{2}}\right]-\frac{V}{2\pi^2}\left(\frac{3MG}{r_0^3}\right)^2\frac{1}{16}\ln\left(\frac{r_0^3\Lambda^2}{3MG}\right)+a\frac{i}{MG}. \tag{64}$$

If we consider only one of the wedges, we obtain the correction relative to the positive (negative) $\mathcal{ADM}$ mass M , depending on the location of the wedge. One can observe that

$$\Delta E\left(M\right)\rightarrow\infty \text{ when } M\rightarrow 0, \text{ for } r_0=2GM \tag{65}$$

and

$$\Delta E\left(M\right)\rightarrow 0 \text{ when } M\rightarrow 0, \text{ for } r_0\neq 2GM. \tag{66}$$

Note that this singular behaviour is independent of boundary conditions since it is related to the volume term.

V. SUMMARY AND CONCLUSIONS

We started from the problem of defining (semiclassical) quantum corrections to the quasilocal energy. By means of a variational approach with Gaussian wave functionals, an attempt to calculate such a correction was made. By construction, we used the subtraction procedure given in Refs. [4–6] to avoid divergences coming from boundaries. Despite the constraint equations, this calculation is based on an extension of the subtraction procedure involving volume terms in the semiclassical regime. Excitations coming from boundary terms have been neglected to avoid the unphysical situation of having contributions deriving from infinity. In this context, the extended subtraction procedure corresponds to the difference between zero point energies calculated in an asymptotically flat background referring to a flat background. This procedure eliminates the UV divergence of the free gravitons, leaving the contribution of the curved background related to an *imposed by hand* UV cut-off. Note that the subtraction procedure at one loop has the correct ingredients to be related to the Casimir energy. This seems to give some information about the vacuum behaviour. Indeed, if we look at symmetric boundary conditions with respect to the bifurcation surface S_0, then eq.(64) can be interpreted as an energy gap measuring the probability of creating a

black hole pair by a topological fluctuation; it also indicates that flat spacetime is unstable with respect to pair creation. In the introduction we mentioned the impossibility of generating single black holes by quantum fluctuation in a flat space having zero temperature, because the energy would not be conserved [2,16]. However, in the neutral black hole pair scenario [3], where each component resides in a different universe, the energy is conserved provided that the boundaries are symmetric with respect to the bifurcation S_0. Note that this calculation can be related to the quantity

$$\Gamma_{1-\mathrm{hole}} = \frac{P_{1-\mathrm{hole}}}{P_{\mathrm{flat}}} \simeq \frac{P_{\mathrm{BlackHolePair}}}{P_{\mathrm{flat}}}. \tag{67}$$

To generalize a little more, suppose to enlarge this process from one pair to a large but fixed number of such pairs, say n. What we obtain is a multiply connected spacetime with n holes inside the manifold, each of them acting as a single bifurcation surface with the sole condition of having symmetry with respect to the bifurcation surface even at finite distance. Let us suppose the interaction between the holes can be neglected, i.e., let us suppose that the total energy contribution is realized with a coherent summation process. This is equivalent to saying that the wave functional support (here, the semiclassical WDW functional) has a finite size depending only on the number of the holes inside the spacetime. It is clear that the number of such holes cannot be arbitrary, but is to be related with a minimum size of Planck's order. Thus, assuming a coherency property of the wave functional and therefore of the N-holes spacetime, eq.(67) has to be generalized to

$$\Gamma_{\mathrm{N-holes}} = \frac{P_{\mathrm{N-holes}}}{P_{\mathrm{flat}}} \simeq \frac{P_{\mathrm{foam}}}{P_{\mathrm{flat}}}. \tag{68}$$

Recall that a hole, here, has to be understood as a wormhole. Then, what eq.(68) suggests is the possibility of generating a *foamy* spacetime, with wormholes as building blocks.

APPENDIX A: CONVENTIONS AND SCALAR CURVATURE EXPANSION

- Riemann tensor

$$R^{l}_{ijm} = \Gamma^{l}_{mi,j} - \Gamma^{l}_{ji,m} + \Gamma^{l}_{ja}\Gamma^{a}_{mi} - \Gamma^{l}_{ma}\Gamma^{a}_{ji}. \tag{A1}$$

- Ricci tensor

$$R_{im} = R^{l}_{ilm}. \tag{A2}$$

- Scalar curvature

$$R = g^{lj}R_{lj}. \tag{A3}$$

- In three dimensions, the Weyl tensor vanishes, then the Riemann tensor is completely determined by Ricci tensor

$$R_{lijm} = g_{lj}R_{im} - g_{lm}R_{ij} - g_{ij}R_{lm} + g_{im}R_{lj}. \tag{A4}$$

- Second order scalar curvature

$$\int d^3x \left[-\frac{1}{4}h\triangle h + \frac{1}{4}h^{li}\triangle h_{li} - \frac{1}{2}h^{ij}\nabla_l\nabla_i h^{l}_{j} + \frac{1}{2}h\nabla_l\nabla_i h^{li} - \frac{1}{2}h^{ij}R_{ia}h^{a}_{j} + \frac{1}{2}hR_{ij}h^{ij} \right]. \tag{A5}$$

APPENDIX B: THE KINETIC TERM

The Schrödinger picture representation of the kinetic term is

$$G_{ijkl}\pi^{ij}\pi^{kl} = G_{ijkl}\left(-\frac{\delta^2}{\delta h_{ij}(x)\,\delta h_{kl}(x)} \right). \tag{B1}$$

We have to apply this quantity to the gaussian wave functional $|\Psi\rangle$. This means that

$$\pi^{ij}(x)\,\pi^{kl}(x)\,|\Psi\rangle = -\frac{\delta^2\Psi[h]}{\delta h_{ij}(x)\,\delta h_{kl}(x)}$$

$$= \frac{1}{2}K^{-1(kl)(ij)}(x,x)\left(\sqrt{g(x)}\right)^2 \Psi[h]$$

$$-\frac{1}{4}\int d^3y' d^3y'' \left(\sqrt{g(x)}\right)^2 \sqrt{g(y')}\sqrt{g(y'')}K^{-1(kl)(k'l')}(x,y')\,h_{k'l'}(y')$$

$$\cdot K^{-1(ij)(k''l'')}\left(x,y''\right)h_{k''l''}\left(y''\right)\Psi\left[h\right]. \tag{B2}$$

By functional integration

$$\left\langle \Psi\left|h_{k'l'}\left(y'\right)h_{k''l''}\left(y''\right)\right|\Psi\right\rangle = K_{(k'l')(k''l'')}\left(y',y''\right)\left\langle \Psi|\Psi\right\rangle. \tag{B3}$$

Then

$$\left\langle \Psi\left|\pi^{ij}\left(x\right)\pi^{kl}\left(x\right)\right|\Psi\right\rangle$$

becomes

$$\frac{1}{2}K^{-1(kl)(ij)}\left(x,x\right)\left(\sqrt{g\left(x\right)}\right)^{2}$$

$$-\frac{1}{4}\int d^{3}y'd^{3}y''\left(\sqrt{g\left(x\right)}\right)^{2}\sqrt{g\left(y'\right)}\sqrt{g\left(y''\right)}K^{-1(kl)(k'l')}\left(x,y'\right)K^{-1(ij)(k''l'')}\left(x,y''\right)$$

$$K_{(k'l')(k''l'')}\left(y',y''\right)\left\langle \Psi|\Psi\right\rangle$$

$$=\frac{1}{4}K^{-1(kl)(ij)}\left(x,x\right)\left(\sqrt{g\left(x\right)}\right)^{2}\left\langle \Psi|\Psi\right\rangle. \tag{B4}$$

Then the expectation value of the kinetic term, with the Planck length reinserted, is

$$\left\langle T\right\rangle = \frac{1}{4l_{p}^{2}}\int d^{3}x\sqrt{g}\left(G_{ijkl}K^{-1(kl)(ij)}\left(x,x\right)\right), \tag{B5}$$

APPENDIX C: SEARCHING FOR NEGATIVE MODES

In this paragraph we look for negative modes of the eigenvalue equation (37). For this purpose we restrict the analysis to the S wave. Indeed, in this state the centrifugal term is absent and this gives the function $V\left(x\right)$ a potential well form, which is different when $l\geq 1$, where l is the angular momentum. However, this potential form is valid only for the H component, i.e.

$$-\triangle H\left(r\right)-\frac{4m}{r^{3}}H\left(r\right)=-E^{2}H\left(r\right),$$

with $E^2 > 0$. Passing to the reduced field $H(r) = \frac{h(r)}{r}$, we obtain

$$-\frac{d}{dr}\left(\sqrt{1-\frac{2m}{r}}\frac{dh}{dr}\right)+\left(\frac{-3m}{r^3}+E^2\right)\frac{h}{\sqrt{1-\frac{2m}{r}}}=0.$$

Making the substitution (43), the equation becomes

$$-\frac{dx}{dr}\frac{d}{dx}\left(\sqrt{1-\frac{2m}{r}}\frac{dh}{dx}\frac{dx}{dr}\right)+\left(-\frac{3m}{r^3}+E^2\right)\frac{h}{\sqrt{1-\frac{2m}{r}}}$$

$$=-\frac{d}{dx}\left(\frac{dh}{dx}\right)+\left(-\frac{3m}{r^3}+E^2\right)h=0,$$

where $\frac{dx}{dr} = \frac{1}{\sqrt{1-\frac{2m}{r}}}$. Near the horizon $x \simeq 4m\sqrt{\frac{r}{2m}-1}$or $r \simeq 2m\left(1+\left(\frac{x}{4m}\right)^2\right)$. Define $\rho = \frac{r}{2m} \Longrightarrow \rho = 1+\left(\frac{y}{2}\right)^2$ with $y = \frac{x}{2m}$. Then we get

$$-\frac{d}{dy}\left(\frac{dh}{dy}\right)+\left(-\frac{3m}{(2m)^3\rho^3(y)}+E^2\right)h\rightarrow-\frac{d^2h}{dy^2}+\left(-\frac{3}{2\left(1+\left(\frac{y}{2}\right)^2\right)^3}+\lambda\right)h=0,$$

where $\lambda = (2m)^2 E^2$. Expanding the potential around $y = 0$, one gets

$$-\frac{d^2h}{dy^2}+\left(-\frac{3}{2}\left(1-\frac{3}{4}y^2\right)+\lambda\right)h=0.$$

We see that this is a quantum harmonic oscillator equation whose spectrum is well known, that is $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$. However, we have to remark that the original spectrum must be contained in the interval , where $-\frac{3}{2}$ represents the bottom of the approximated potential. Then $E_n = \frac{3}{2}+\lambda_n = \sqrt{\frac{3}{8}}\left(n+\frac{1}{2}\right)$, where $\omega = \sqrt{\frac{3}{8}}$ and $\hbar = 1$ in natural units. Thus

$$\lambda_n = -\frac{3}{2}+\frac{3}{2}\frac{\sqrt{2}}{2}\left(n+\frac{1}{2}\right).$$

We see that

$$\lambda_0 = -0.975 \text{ and } \lambda_1 = -\frac{3}{2}(-0.06) \Longrightarrow \lambda_1 \notin \left(-\frac{3}{2},0\right).$$

There is only **one eigenvalue.** The same problem can be approached with the Rayleigh-Ritz method along a numerical integration and the result is $\lambda_0 = -1.094$.

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